Topology and its Applications 156 (2009) 1749–1751

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# New basic result in classical descriptive set theory: Preservation of completeness

Alexey Ostrovsky *

*Helmut-Käutner Str. 25, 81739 Munich, Germany*

ARTICLE INFO

*Article history:*
Received 23 September 2008
Received in revised form 26 February 2009

*MSC:*
primary 03E15, 54C10, 54H05
secondary 54E40, 26A21

*Keywords:*
Borel space
Completely metrizable space
Open map
Closed map
s-Covering map
Stable map

ABSTRACT

The aim of this note is to prove the following result:
Assume that $f$ is a continuous function from the space of irrationals $\omega^\omega$ onto $Y$ such that the image $f(U)$ of every set $U$ which is open and closed in $\omega^\omega$ is the union of one open and one closed set. Then $Y$ is a completely metrizable space.



All spaces in this paper are supposed to be separable and metrizable and all the maps $f : X \to Y$ to be continuous and onto. By $\omega^\omega$ we denote the space of irrationals. A space $X$ is called an *$F_{II}$-space* if the following conditions hold: $X$ does not contain any countable perfect subset $X'$ (all such $X'$ are homeomorphic to the space of rationals $\mathbb{Q}$). A clopen set is a set which is both open and closed.

A long time ago Sierpiński and Vainštain have established that if $Y$ is an image of a completely metrizable space $X$ under a closed or an open map, then $Y$ is completely metrizable [13,14].

There have been numerous attempts to define the common class of maps that would contain all the classes of open and closed maps with their good common property: preservation of complete metrizability.

The first such class – the class of s-covering maps[1] – was introduced and investigated in [6,7].

The most broad class – the class of stable maps[2] – was discovered by the author 20 years later [11].

The class of stable maps includes compositions of open or closed maps [11], which is not the case for the class of s-covering maps [10, Example].

Note that stable maps (closed, open, s-covering maps) of separable metric spaces preserve $F_{II}$-spaces [11, Theorem 1] (see also [6]).

* Tel.: +49896256676.
*E-mail address:* Alexey.Ostrovskiy@UniBw.de.

[1] The map $f : X \to Y$ is *s-covering* if for every countable compact $K \subset Y$ there is a compact $B \subset X$ for which $f(B) = K$.

[2] The map $f : X \to Y$ is *stable* if for every $y \in Y$ there is a nonempty family $\eta_y$ of open subsets of $X$ intersecting $f^{-1}(y)$ such that for every $U \in \eta_y$ and every open $V \supset U \cap f^{-1}(y)$ there is an open neighborhood $O(y)$ of $y$ such that $V \in \eta_{y'}$ for every point $y' \in O(y)$.

0166-8641/$ – see front matter 
doi:10.1016/j.topol.2009.03.001

Eventually, we shall note that if $f : X \to Y$ is stable, and $X = \omega^\omega$, then there is a subset $Z \subset X$ such that $f(Z) = Y$ and for every clopen (relative to $Z$) subset $U \subset Z$ its image $f(U)$ is $F_\sigma$ set in $Y$ [11, Theorem 0]. Surprisingly, this statement is true for s-covering maps without any limitations on $X$ or on fibers $f^{-1}(y)$ [12, Corollary 3].

The following Theorem 1 and Corollary 4 give a new natural common generalization of Sierpiǹski's and Vainštain's results. In contrast to all previous works the map $f$ in Theorem 1 can be not quotient. It is also rather surprising that the assertion of Theorem 1 was not known even if every $f(W) = V$ is closed in $Y$.

**Theorem 1.** *Let $f : X \to Y$ be a map from $X \subset \omega^\omega$ with the following property:*

($*$) *if $W \subset X$ is clopen in $X$ then $f(W) = U \cup V$ is the union of the sets $U$ and $V$, an open set and a closed set respectively.*

*If $X$ is completely metrizable, then $Y$ is also completely metrizable.*

Theorem 1 is an obvious corollary of Theorem 2 below. Indeed, first note that $Y$ is an absolute Borel space. This follows from the condition ($*$) using [8] (see also [9,4]). In facts, we conclude from ($*$) that $f$ is an open-Borel map[3] i.e. the image $f(U)$ of every open subset $U \subset X$ is a Borel subset of $Y$, hence by [8, Lemma 1] that there is an absolute Borel space $Z \subset X$ such that $f|Z$ is one-to-one map onto $Y$, and finally that $Y$ is an absolute Borel space according to the classical theorem of descriptive set theory.

Further, by Hurewicz's theorem [3] an absolute Borel space is completely metrizable if and only if it is an $F_{\mathrm{II}}$-*space*. Now we must only prove that $Y$ is an $F_{\mathrm{II}}$-space:

**Theorem 2.** *Let $f : X \to Y$ be a map from $X \subset \omega^\omega$ with the property ($*$).*
*If $X$ is an $F_{\mathrm{II}}$-space, then $Y$ is also an $F_{\mathrm{II}}$-space.*

We need to prove the following lemma (the closure of $V$ in $X$ will be denoted by $[V]_X$):

**Lemma 3.** *Let $f : X \to Y$ be a map from $X \subset \omega^\omega$ onto a space $Y$ without isolated points with the property ($*$). Denote $V = \bigcup_{y\in Y} \operatorname{Int} f^{-1}(y)$ and $W = [V]_X$. Then $f(W)$ is closed in $Y$ and the restriction $f|W$ is a closed map.*

**Proof.** 1. First of all we prove that $f(W)$ is a closed subset in $Y$. Indeed, if $f(W)$ is not closed, then take $y \in Y \setminus f(W)$ and $y_i \to y$, where $y_i \in f(W)$. Since $V$ is dense in $W$ and $f$ is continuous, we can suppose that $y_i \in f(V)$.

Take the clopen sets $O_i \subset f^{-1}(y_i) \cap V$, $\operatorname{diam} O_i < 1/i$. It is easy to see that $M = \bigcup O_i$ does not have any limit points in $f^{-1}(y) \cap W$ since, obviously, $[M]_W \subset [V]_X = W$ and $W \cap f^{-1}(y) = \emptyset$. Hence, $M$ is a clopen set in $X$. By ($*$), $f(M) = \{y_i\}$ is the union of the sets $U$ and $V$, an open set and a closed set respectively. Since $Y$ is without isolated points, $U = \emptyset$, hence, $f(M)$ can only be closed in $Y$. But the set $f(M)$ is not closed in $Y$, since its limit point $y \notin f(M)$.

2. We will prove that $f|W : W \to f(W)$ is a closed map. Obviously, it is equivalent to the following statement:

If $y_i \to y$, where $y, y_i \in f(W)$, then every sequence of points $x_i \in f^{-1}(y_i) \cap W$ has a limit point in $f^{-1}(y) \cap W$.

Suppose the opposite, then there are $y, y_i \in f(W)$, $y_i \to y$ and a sequence of points $x_i \in f^{-1}(y_i) \cap W$ which have no limit point in $f^{-1}(y) \cap W$.

Since $V$ is dense in $W$ and $f$ is continuous, we can take $x'_i \in V$, $\operatorname{dist}(x'_i, x_i) < 1/i$ and $\operatorname{dist}(f(x'_i), f(x_i)) < 1/i$. Denote $y'_i = f(x'_i)$, then $y'_i \to y$.

Choose the clopen sets $O(x'_i) \subset f^{-1}(y'_i) \cap V$ such that $\operatorname{diam} O(x'_i) < 1/i$.

It is easy to see that $M' = \bigcup O(x'_i)$ has no limit point in $f^{-1}(y) \cap W$ and is a clopen set in $X$.

By supposition, $f(M') = \{y'_i\}$ is the union of the sets $U$ and $V$, an open set and a closed set respectively. Since $Y$ is without isolated points, $U = \emptyset$, hence, $f(M')$ can only be closed in $Y$.

On the other hand, $f(M')$ is not closed in $Y$, since its limit point $y \in f(W) \setminus f(M')$. □

We now return to the proof of Theorem 2.

If $Y$ is not an $F_{\mathrm{II}}$-space then it contains a countable perfect subset $Q$, which is homeomorphic to the space of rationals $\mathbb{Q}$.

**Remark.** The map $f|f^{-1}(Q)$ has also property ($*$). Indeed, let $U_1 \subset f^{-1}(Q)$ be a clopen set in $f^{-1}(Q)$, then $U_2 = f^{-1}(Q) \setminus U_1$ is open and closed in $f^{-1}(Q)$ too. For every point $x \in U_1$ the distance $\operatorname{dist}(x, U_2)$ is a positive number and we can consider a ball $U(x)$, which is clopen in $X$ and $\operatorname{diam} U(x) < \operatorname{dist}(x, U_2)$. Analogously, take for every $x \in U_2$ a clopen ball $U(x)$ such that $\operatorname{diam} U(x) < \operatorname{dist}(x, U_1)$. Finally, we can take (since $f^{-1}(Q)$ is closed in $X$) for every $x \in X \setminus f^{-1}(Q)$ a clopen ball $U(x) \subset X \setminus f^{-1}(Q)$.

[3] Open-Borel map is called OB-map in russian version of [8].

Since $X \subset \omega^\omega$, $X$ is Lindelöf space and by [5, §26, II, Theorem 1] there is an open refinement of the open cover $\{U(x)\colon\ x \in X\}$ from pairwise disjoint sets $V_\alpha$, which are clopen in $X$ and such that every $V_\alpha$ lies in some $U(x)$. The union of all sets $V_\alpha$, which intersect $U_1$ is denoted by $V_1$. It is clear that it is clopen in $X$ and $V_1 \cap f^{-1}(Q) = U_1$.

Hence, $f(U_1) = Q \cap f(V_1)$ and we can suppose $Y = Q$.

According to Lemma 3 if $T = Q \setminus f(W) \neq \emptyset$ then $T = Q \setminus f(W)$ is an open subset in $Q$ and hence $f^{-1}(T)$ is an $F_{\mathrm{II}}$-space (as open subset of the $F_{\mathrm{II}}$-space $X$). But this is impossible because all the sets $f^{-1}(y)$, $y \in T$ are nowhere dense in $X$ and in $f^{-1}(T)$ and $T$ is countable.

Hence, $T = Q \setminus f(W) = \emptyset$ and $f|W$ is a closed map onto $Q$. This contradicts our assumption that $X$ (and hence $W$) is completely metrizable and that closed maps preserve $F_{\mathrm{II}}$-spaces.

**Corollary 4.** *Let $F : Z \to Y$ be a map from a (not necessarily 0-dimensional) separable metric space $Z$ with the following property:*

*for every open or for every closed subset $W \subset Z$ its image $F(W) = U \cup V$, where $U$ is open and $V$ is closed in $Y$.*

*If $Z$ is completely metrizable, then $Y$ is also completely metrizable.*

Indeed, a separable metric space $Z$ is complete if and only if it is a continuous image of the irrationals under an open map $h$ [2] and under a closed map $g$ [1]. It is easily proved that compositions $F \circ g : \omega^\omega \to Y$ and $F \circ h : \omega^\omega \to Y$ have the property $(*)$, hence, by Theorem 1 $Y$ is completely metrizable.

**Question.** I do not know whether the conclusion of Theorem 1 is still true if the condition "$f(W) = U \cup V$" would be replaced by "$f(W) = U \cap V$" or by another combination of open and closed sets.

## References

[1] R. Engelking, On closed images of the space of irrationals, Proc. Amer. Math. Soc. 21 (3) (1969) 583–586.
[2] F. Hausdorff, Ueber innere Abbildungen, Fund. Math. 23 (1934) 279–291.
[3] W. Hurewicz, Relativ perfekte Teile von Punktmengen und Mengen (A), Fund. Math. 12 (1928) 78–109.
[4] K. Kuratowski, A. Maitra, Some theorems on selectors and their applications to semicontinuous decompositions, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. 22 (1974) 887–891.
[5] K. Kuratowski, Topologie I, Academic Press, New York, 1966.
[6] A. Ostrovskii, On *k*-covering mappings of separable metric spaces, Soviet Math. Dokl. 13 (1972) 289–291, MR0296916 (45 #5975).
[7] A.V. Ostrovskii, On compact-covering mappings, Soviet Math. Dokl. 17 (1976) 606–610, MR0402702 (53 #6518).
[8] A.V. Ostrovsky, Borel extensions, selectors and nonisomorphic A-sets, in: Proceedings of the Leningrad International Topology Conference, 1982, pp. 84–90 (in Russian); Leningrad International Topological Conference, Abstracts, 1982, p. 119.
[9] A.V. Ostrovskii, On the Luzin–Yankov theorem and the general theory on selections, Russian Math. Surveys 40 (1985) 226–227, MR0786101 (86g:54030).
[10] A.V. Ostrovskii, New classes of maps related to covering maps, Moscow Univ. Math. Bull. 49 (1994) 20–23, 75, MR1317090 (96a:54013).
[11] A. Ostrovsky, Stable maps of polish spaces, Proc. Amer. Math. Soc. 128 (10) (2000) 3081–3089, MR1664430 (2000m:54017).
[12] A. Ostrovsky, s-Covering maps with complete fibers, Topology Appl. 102 (2000) 1–11.
[13] W. Sierpiǹski, Sur une propriete des ensembles $G_\delta$, Fund. Math. 16 (1930) 173–180.
[14] I.A. Vainštain, On closed mappings, Moskov. Gos. Univ. Uč. Zap. 155 (5) (1952) 3–53 (in Russian).